Department of Mathematics

Pattamundai College, Pattamundai

4^{th} Semester

Riemann Integration and Series of Functions

Core – 9 (Analysis – III)

Sec–A

(Unit–1)

1. Define the norm of a partation.
2. What is common refinement.
3. Show that L (p, –f) = – U (p.f).
4. If f : [a,b] $\to$ R is a bounded function and p,$p^1 \in$ p[a,b] such that $P \subset P^1$. then show that $\cup$(p.f) $\geq \cup(p^1.f)$.
5. Let f(x) = x^2 for $x \in [0,1]$ and let P = { 0, $\frac{1}{3}$, $\frac{2}{3}$,1 } be a partation of [0,1] compute U(p.f) and L (p.f).
6. If f : [a,b] $\to$ R is bounded function, then m (b–a) $\leq \int_a^b f(x)dx \leq f\int_a^b f(n)\,dx \leq$ M (b–a). (T/F)
7. Write the statement of Darbouxs theorem for Riemann integration.
8. If the set of points ofdiscontinuitu of a bounded function f : [a,b]$\to$R has finite number of limit points then f is integrable on [a,b]. (T/F)
9. If |f(x)| is not integrable then f is not integrable. (T/F)
10. Let f, g$\in$R [a,b] and f(x) $\leq$ g(x) for all x $\in$ [a,b]. Then show that $\int_a^b f(x)\,dx \leq \int_a^b g(x)dx$.
11. Find the limit, $\lim_{n\to\infty}\left[n \sum_{k=1}^{n} \frac{1}{(n+k)^2}\right]$
12. Find the limit, $\lim_{n\to\infty}\left(\sum_{r=1}^{n} \frac{1}{(2n+r)}\right)$

(Unit–2)

13. What is proper integral?
14. What is improper integral?
15. Give an example of first kind of improper integral.
16. Give an example of second kind of improper interal.
17. Give an example of third kind of improper integral.
18. Examine the convergence of improper integral $\int_1^\infty \frac{1}{x}dx$.
19. Examine the convergence of the integral $\int_0^1 \log x\,dx$

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20. Let f and g be two positive functions on (a, b), a being the only point of infinite discontinuity. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ converges $\Rightarrow \int_a^b f(x)dx$ is ______________.

21. Find the value of $\Gamma\left(\frac{7}{2}\right)$, where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

22. $B(m, n) = \frac{\Gamma(m).\Gamma(n)}{2\Gamma(m+n)}$ is True or False.

23. $\int_0^1 \sqrt{x}\,(1-x)^3\,dx$, write in the form of Beta function.

24. Beta function is a symmetric function?

(Unit–3)

25. Write the difference between the pointwise convergence and uniform convrgence.

26. What is limit function of the sequence of function $f_n(x)$.

27. Find the limit function of $f_n(x) = x^n$, $x \in [0,1]$.

28. Write the geometrical interpretation of uniform convergence.

29. Write only the statement of Dedekind's test for uniform convergence of a series.

30. Show that the series $\sum_{x\to 1}^{\infty} \frac{\sin nx}{n^{\alpha}}$ is uniformly and absolutely convergent for $\alpha > 1$.

31. Write only the statement of weierstrass M-test.

32. Show that the series $\sum_{n\to 1}^{\infty} \frac{\cos x}{n^{p}}$ is uniformly and absolutely convergent for all real number x and P>1.

33. If the series $\Sigma\, a_n$ converges absolutely then prove that $\Sigma\, a_n \cos nx$ and $\Sigma\, a_n \sin nx$ are uniformly convergent on R.

34. Write only the statement of Abel's test for uniformly convergence of the series.

35. Prove that the series $\Sigma\, \frac{(-1)^{n-1}}{n} x^n$ is uniformly conversent on [0,1].

36. If $\Sigma\, a_n$ is convergent, then show that $\Sigma\, \frac{a_n}{n} x$ is uniformly convergent on [0, 1]

37. Give an example of uniformly conversent serie but not absolutely convergent.

38. Give an example of a sequence of integrable functions converging to a non-integrable function.

39. Give an example of a squence of convergent but not uniformly convergent continuous functions with limit function being continuous.

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40. Give an example of differentiable functions converging to a non-differentiable function.

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(Unit–4)

41. Find the radius of convergence of the series $\sum_{n=0}^{\infty} n!\, x^n$.

42. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n}}{(2n)!}$

43. Find the radius of conversence of the series $\sum_{n=0}^{\infty} \frac{(n+1)^2}{(n^2+2n+5)^3} x^n$.

44. Suppose a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence is R and r<R then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent in the interval (o,r). (T/F).

45. Suppose a power series $f(x) = \sum_{n=0}^{\infty} 3^n x^n$ then find $f^1(x)$ = ________?

46. Suppose a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then find 3x.f(x) = ________?

47. Let Σ an, Σ bn be convergent, where cn is given by $cn = a_{o}bn + a_1 b_{n-1} + ... + a_n b_o$

Then $(\sum_{n=0}^{\infty} an)(\sum_{n=0}^{\infty} bn) = \sum_{n=0}^{\infty} cn$. (T/F)

48. Show that $\sum_{n=0}^{\infty} (\frac{1}{2})^n = 2$

49. Suppose a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $\frac{1}{R} = \lim_{n\to\infty} \text{Sup} \sqrt[n]{|an|}$ is called as Hadamard formula. (T/F)

50. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges in |x| < R, then derivatives of all orders exist in |x| < R and $f^k(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)...(n-k+1)\, an\, x^{n-k}$, K = 0,1, 2,3...... (T/F).

Sec–B

(Unit–1)

1. If f : [a,b] $\to$ R is a bounded function and P$\in$P[a,b], then $m(b-a) \le L(p,f) \le \cup(p.f) \le M(b-a)$.

2. If f : [a,b]$\to$ R, g : [a,b] $\to$ R are bounded function and P$\in$P [a,b], then $U(p,f+g) \le U(p,f) + U(p,g)$.

3. If P' is a refinement of P containing P points more than P and $|f(x)| \le k$ for all x $\in$[a,b], then w(p.f) – w(p,f) $\le 4pk\delta$, where $||P|| = \delta$.

4. If f : [a,b] $\to$ R is a bounded function, then $\underline{\int_a^b} f(x)dx \le \int_a^{\overline{b}} f(x)dx$.

5. Compute L (p.f) and U (p.f) for the function f defined by $f(x) = x^2$ on [0,1] and $p = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1,\}$

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6. Show by an example that every bounded function need not be R- integrable.

7. If f: [a,b] $\to$ R is monotonic on [a,b] then f is integrable on [a,b].

8. If the set of points of discontinuity of a bounded function f : [a,b] $\to$ R is finite, then f is integrable on [a,b].

9. If f is integrable on [0,1], then $\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

10. Evaluate $\int_{-1}^{2} f(x)\, dx$, where f(x) = |x|.

11. If f $\in$ R [a,b] then $f^2 \in$ R [a,b].

12. Let f be defined by $f(x) = \begin{cases} \sqrt{1-x^2}, & x \in Q \cap [0,1] \\ 1-x & x \in Q^c \cap [0,1] \end{cases}$ Then show that $\int_0 f(x)da = \frac{\pi}{4}$, $\int_0^1 f(x)\, dx = \frac{1}{2}$

(Unit–2)

13. Show that $\int_1^\infty \frac{x}{(1+2x)^3}\, dx = \frac{5}{72}$

14. Show that $\int_{-\infty}^{\infty} \frac{dx}{(1+n^2)^2}\, dx = \frac{\pi}{2}$

15. Show that $\int_0^{1/e} \frac{dx}{x(\log x)^2}\, dx = 1$

16. Show that $\int_0^1 \frac{dx}{x^3(2+x^2)^5}$ is divergent.

17. Find the value of m and n for which the integral $\int_0^1 e^{-mx} x^n\, dx$ converges.

18. Show that $\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{\frac{1}{3}}}\, dx$ is divergent.

19. Show that $\int_a^t f(x)\, dx$ is bounded for all t $\geq$ a and g(x) is a bounded and monotonic function for x$\geq$a, tending to 0 as x$\to\infty$ then $\int_\infty^a f(x).\, g(x)\, dx$ is convergent at ∞.

20. Show that B(m,n) = B (n,m)

21. Prove that $\int_0^a (a-x)^{m-1} x^{n-1}\, dx = a^{m+n-1}$ B(m,n).

22. Prove that $\int_0^2 x^4(8-x^3)^{-1/3}\, dx = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$

23. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

24. Show that $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$

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(Unit–3)

25. Let (r_n) be a sequence consisting of all the rational numbers and for n =1, 2, 3...... define the functions f_n on R by $f_n(x) = \begin{cases} 1 & (x = r_n) \\ 0 & \text{otherwise} \end{cases}$ prove that fn convergence pointwise but not uniformly on every interval of R.

26. Show that the sequence $<f_n(x)>$ where $f_n(x) = \tan^{-1}(nx)$ is not uniformly convergent on [0,1].

27. If a series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to f(x) on [a,b] and each function $f_n(x)$ is integrable on [a, b], the f is integrable on [a,b] and $\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx$ converges uniformly to $\int_a^b f(x)dx$.

28. Show that the sequence <fn>, where $fn(x) = nx\,e^{-nx^2}$, $n \in N$ is not uniformly conversent on [0,1].

29. Show that the sequence <fn>, where $fn(x) = \frac{nx}{1+n^2x^2}$, $0 \le x \le 1$ can not be differentiable term by term at x =0.

30. Suppose that the series Σ fn(x) converges uniforms to f(x). Then prove that $\sum_{n=0}^{\infty}$ fn(x) g(x) converges uniformly to f(x). g(x) for every bounded function g.

31. Find the pointwise limit of fn(x) if it exists, then determine whether fn(x) converges uniformly or not where $fn(x) = n^2x(1-x^n)$, $x \in [0,1]$.

32. If $\Sigma f_k(x)$ and Σ gk(x) are uniformly convergent then prove that $\Sigma (f_k(x) + g_k(x)$ is uniformly convergent.

33. Verify the function of sequence is uniformly convergent or not, where $fn(x) = nx\,e^{-nx^2}$, $x \in [0,1]$

34. Verify the function of sequence is uniformly conversent or not fn(x) dx $= \frac{nx^3}{1+nx}$, $x \in [0, 100]$

35. Let <fn> be a sequence of continuous functions on [a,b] and suppose $fn \to p$ uniformly on [a,b]. Then prove that $\lim_{n\to\infty}\int_a^b fn(x)\,dx = \int_a^b \lim_{n\to\infty} fn^{(x)}\,dx$

36. Let <fn> be a sequence of functions in R [a,b] converging uniformly to f. Then show that $f \in R$ [a,b] and $\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

37. Let $fn(x) = \frac{1}{1+nx}$, $0 \le x \le 1$, show that $\lim_{n\to\infty}\int_0^1 fn(x)\,dx = \lim fn(x)\,dx$ even if the convergence of fn is not uniformly.

38. Examine the pointwise and uniform convergence of the series $\sum_{n=0}^{\infty}\frac{x}{n(1+nx^2)}$ on R.

39. Examine the pointwise and uniform convergence of the series $\sum_{n=0}^{\infty}\frac{x^n}{1+x^n}$

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40. Examine the pointwise and uniform convergence of the series $\sum_{n=0}^{\infty} \frac{x}{n(1+x)}$

(Unit – 4)

41. If the power series $\sum_{n=0}^{\infty} an\, x^n$ has radius of convergence r, then show that the power series $\sum_{n=1}^{\infty} n^{an} x^{n-1}$, $\sum_{n=0}^{\infty} \frac{an}{n+1} x^{n+1}$ have also the radius of convergence r.

42. If $\Sigma\, a_n x^n$ and $\Sigma\, b_n x^n$ both have radius of convergence r>0 and if $\Sigma\, b_n x^n = \Sigma\, a_n x^n$ for |x|<r, then prove that an=bn for each n.

43. Let Σ an and Σ bn be convergent where cn is given by the cauchy product, $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n$ where cn = $a_0$bn + $a_1$bn + +....+ $a_n b_0$

44. Show that for |x|<1, $\alpha \in R$, then show that $(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{n}{k} x^k = 1 + \alpha^{x} + \frac{\alpha(\alpha-1)}{2l} x^2 + \ldots..$

45. Show that if Σ an converges, then $\Sigma \frac{an}{n+1}$ converges and $\int (\sum_{n=0}^{\infty} an\, x^n)\, dx = \sum_{n=0}^{\infty} \frac{an}{n+1}$

46. Prove that $f(x) = 1-x^2+x^3-x^5+x^6-x^8+x^9-x^{11}+\ldots..$ has radius of convergence 1 and show that $\lim_{x\to 1} f(x) = 2/3$

47. Show that $e^{-x^2} = \sum_{n=1}^{\infty} \frac{(-1)}{n!} x^{2n}$, $x \in R$

48. For every $x \in R$ and $n \geq 0$, then prove that $\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$

49. Find the sequence of Bernstein's Polynomials of f(x) = x.

50. Find the sequence of Bernstein's Polynomials of $f(x) = x^2$.

Sec–C

(Unit–1)

1. If f is defined on [0,1] by f(x) = x ∀ x ∈[0,1] then f∈R [0,1] and show that $\int_0^1 f(x)\, dx = \frac{1}{2}$

2. A bounded function f is integreable on [a,b] if and only if for each ∈>0. there exists a partation p of [a,b] such that lU(p.f) – L (p.f)l<∈.

3. Show that a function f defined on [0,1] by $f(x) = \begin{cases} \frac{1}{n} & \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0 & x = 0 \end{cases}$, n = 1, 2, 3...... is integrable on [0,1]. Also show that $\int_0^1 f(x) = \frac{\Pi^2}{6} - 1$

4. If f ∈ R [a,b] and k∈R, then kf∈R [a,b] and $\int_a^b (kf)\, dx = k \int_a^b f(x)\, dx$.

5. If f ∈ R[a,b] and a<c<b then f∈R [a,c], f∈R [c,b] and $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

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(Unit–2)

6. The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if n <1.

7. Show that $\int_0^{\pi/2}$ sinx log (sinx) dx is convergent with value log $\left(\frac{2}{e}\right)$

8. Show that $\int_0^{\infty} \frac{\sin x}{x}$ dx is convergent.

9. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}$ dx exists if and only if m and n are both positive.

10. Prove that $\Gamma(n) = (n-1)\,\Gamma(n-1)$, when n>1 and also prove $\Gamma(n+1) = n!$, where $n \in N$

(Unit–3)

11. State and prove Cauchy's criterion for uniform covergence.

12. Show that the sequence $<f_n(x)>$, where $fn(x) = x^n$ is uniform convergent on [o,k], K<1 but only point wise convergent on [0,1].

13. Let $f_n^{(x)}$ be a sequence of continuous function of E converging uniformly to f of E. Then show that f is continouous on E.

14. Show that the sequence of function $<f_n^{(x)}>$ where $f_n^{(x)} = \frac{n^2x}{1+n^2x^2}$ is non uniformly convergent on [0,1].

15. Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly convergest in (0,b) b>0 but is not in $(0,\infty)$.

(Unit–4)

16. Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence 1. If the series converges at 1, then show that $\lim_{x\to 1} f(x) = f(1)$. If it converges at –1, $\lim_{x\to -1} + f(x) = f(-1)$.

17. State and prove stirling's formula.

Or

Prove that $\lim_{n\to\infty}\left(\frac{n!\ e^n}{n+\frac{1}{2}}\right) = \sqrt{2\Pi}$ that is $n! \sim \sqrt{2n\Pi}\,.\,n^n.\ e^{-n}$, $n \in N!$

18. Let f: [0,1] $\rightarrow$ R be continuous. Then there exists a sequence of polynomials, the Bernstein polynomials $B_n f \rightarrow f$ uniformly on [0,1].

Or

State and prove weierstrass approximation theorem.

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19. Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $x \in R$ has a radius of convergence r>0. Then show that

a) f is differentiable on (–r, r) and for $x \in (-r, r)$, $f(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$.

b) If f is integrable over (–r, r) and $\int_0^{\infty} f(f)\, dt = \sum_{n=0}^{\infty} \frac{an}{n+1} x^{n+1}$. where $0 \le x \le r$.

20. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{5} .... = \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n+1}}{(2n+1)}$ for $-1 \le x \le 1$ and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

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